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A SIMPLE DERIVATION OF

THE POISSON DISTRIBUTION

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A Simple Derivation of the Poisson Distribution

One of the most important stochastic processes is the Poisson process, in which it is assumed that (a) the numbers of events occurring in nonoverlapping time intervals are independent; (b) the probability of one event's occurring during time dt is $\lambda dt + o(dt)$, where λ is a constant, while the probability that two or more occur is $o(dt)$. Various approaches [1] are known which lead to the result that the probability that n events occur in time t, $p_n(t)$, is

$$p_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!} \qquad (n = 0, 1, 2, \ldots), \tag{1}$$

that is, the Poisson distribution with mean λt. An extremely simple and straightforward derivation of this formula, based on an idea of G. Morant [2], is as follows:

The probability for no event to occur in time t is

$$p_0(t) = \lim_{dt \to 0} \left[1 - \lambda dt - o(dt)\right]^{t/dt} \tag{2}$$

$$= e^{-\lambda t} .$$

Eq. (2) merely expresses the fact that for no event to occur in time t, none may occur in any of the subintervals of length dt into which t may be divided. We now use this result to aid in obtaining Eq. (1).

Let us consider n small nonoverlapping time intervals dt_1, dt_2, ..., dt_n, contained within the time interval $(0,t)$. The probability that n events occur—the first at time t_1 within the interval dt_1, the second at time t_2 within the interval dt_2, and so on—is asymptotically equal to

$$e^{-\lambda t_1} \lambda dt_1 \, e^{-\lambda(t_2-t_1)} \lambda dt_2 \ldots \lambda dt_n \, e^{-\lambda(t-t_n)} ,$$

which reduces to

$$e^{-\lambda t} \lambda^n \, dt_1 \, dt_2 \ldots dt_n .$$

This obtains since no event occurs from time 0 to time t_1; one event occurs in the interval dt_1; no event occurs from time t_1 to t_2; and so on. Hence $p_n(t)$, which is the integral of this expression over all t_n satisfying

$$0 \le t_1 \le t_2 \le \ldots \le t_n \le t ,$$

is given by

$$p_n(t) = e^{-\lambda t} \lambda^n \int_0^t \int_0^{t_n} \ldots \int_0^{t_2} dt_1 \, dt_2 \ldots dt_n \quad (n = 1,2,\ldots) , \tag{3}$$

which immediately yields Eq. (1), since the integral in Eq. (3) equals $t^n/n!$.

Thus, using only the simplest kind of reasoning from probability theory, we have deduced the Poisson distribution from the basic assumptions (a) and (b). Consequently, the need for viewing the Poisson distribution as a limiting case of some other distribution is obviated. In addition the technique used readily generalizes to the case in which λ depends on t.

References

[1] W. Feller, An Introduction to Probability Theory and Its Applications, John Wiley and Sons, New York, 1950.

[2] G. Morant, "On Random Occurrences in Space and Time, When Followed by a Closed Interval," Biometrika, Vol XIII (1921), pp 309-337.